

The spectrum of anomalous magnetohydrodynamics

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Abstract

The equations of anomalous magnetohydrodynamics describe an Abelian plasma where conduction and chiral currents are simultaneously present and constrained by the second law of thermodynamics. At high frequencies the magnetic currents play the leading role and the spectrum is dominated by two-fluid effects. The system behaves instead as a single fluid in the low-frequency regime where the vortical currents induce potentially large hypermagnetic fields. After deriving the physical solutions of the generalized Appleton-Hartree equation, the corresponding dispersion relations are scrutinized and compared with the results valid for cold plasmas. Hypermagnetic knots and fluid vortices can be concurrently present at very low frequencies and suggest a qualitatively different dynamics of the hydromagnetic nonlinearities.

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1 Introduction

Electrically conducting media are customarily described as a single fluid in the low-frequency branch of the plasma spectrum. This approach has been extensively applied to the analysis of hydromagnetic nonlinearities [1] evolving in terrestrial [2] and astrophysical plasmas [3, 4, 5]. The same strategy cannot be extended to higher frequencies where the one-fluid description is no longer tenable [6] and the plasma must be treated, at least, as a double fluid. This well known aspect of conventional electromagnetic plasmas stems directly from the properties of the vector currents which are associated, in the high-frequency limit, with the ions and with the electrons. When the plasma is globally neutral the total vector current is instead Ohmic in the low-frequency domain.

A problem of similar nature occurs in anomalous magnetohydrodynamics [7] describing a charged fluid where axial and vector currents are simultaneously present: while the axial currents are not conserved because of the triangle anomaly, the vector currents are eventually Ohmic. The purpose of this investigation is a systematic discussion of the spectrum of anomalous magnetohydrodynamics (AMHD in what follows). The equations of AMHD differ from the ones where only chiral currents are present [8, 9] at finite fermionic density. They generalize the system firstly explored in Ref. [10] accounting for the evolution of the hypermagnetic and hyperelectric fields in the electroweak plasma. Indeed, in the symmetric phase of the electroweak theory the non-screened vector modes of the plasma correspond to the hypercharge which has a chiral coupling to fermions. The axial currents may be associated with the evolution of the chemical potential or with the presence of an axionlike field [11, 12] (see also [13]). In both cases the plasma may host parity-odd configurations of the gauge fields characterized non-vanishing hypermagnetic gyrotropy $\vec{B} \cdot \vec{\nabla} \times \vec{B}$ which is the hydromagnetic analog of the kinetic gyrotropy (i.e. $\vec{v} \cdot \vec{\nabla} \times \vec{v}$) naturally appearing in the discussion of mean-field dynamos [1, 4]. The dynamical production of hypermagnetic knots and Chern-Simons waves during inflation offers a potentially viable mechanism for the generation of the baryon asymmetry of the Universe (see last two papers in [10]). In AMHD the hypermagnetic current is complemented by a vortical current possibly leading to the formation of fluid vortices.

The same class of physical systems previously discussed in the electroweak plasma also arises in the framework of the so called chiral magnetic effect [14]. Both phenomena are often presented as macroscopic manifestations of triangle anomalies. The model of chiral liquid emerging in the context of AMHD could then be relevant also in the context of the chiral magnetic effect insofar as axial currents and quark vector currents are concurrently present in the strongly interacting plasma. In the absence of finite conductivity effects (see e.g. [15, 16]) the validity of the second law of thermodynamics is guaranteed by the simultaneous presence of an hypermagnetic current and of a chiral vortical term. In AMHD the vector currents (eventually responsible of Ohmic dissipation), the chiral currents (determining the anomalous effects) and the vortical currents (required by the second principle of thermodynamics) are

all described by the appropriate kinetic coefficients. Whenever possible AMHD will be discussed in analogy with the spectrum of conventional plasmas. More specifically the plan of this investigation is the following. In section 2 we discuss the relativistic problem and derive the general form of the kinetic coefficients. In section 3 the properties of the two-fluid equations are analyzed while section 4 is devoted to the dispersion relations in the high-frequency domain. The one-fluid equations and their implications are presented in section 5. Section 6 contains our concluding remarks. To avoid digressions some relevant technical aspects have been relegated to the appendix.

2 The relativistic formulation and the total entropy

The conservation of the total energy momentum tensor and the evolution of the chiral and vector currents determine the relativistic form of the second law of thermodynamics. If the four-divergence of the entropy four-vector is to be positive semi-definite (as implied by the generalized second law) the chiral and vector currents must contain supplementary kinetic coefficients corresponding to the hypermagnetic and to the vortical currents. In what follows the dissipative effects are included in the framework of the Landau approach: the total four-velocity coincides then with the velocity of the energy transport defined from the mixed components of the total energy-momentum tensor.

2.1 Ohmic and chiral currents

In the simplest situation the total energy-momentum tensor of the system ($T_{\mu\nu}^{(tot)}$ in what follows) consists of four qualitatively different contributions: the energy-momentum tensor of the charged species (denoted by $T_{\mu\nu}^{(\pm)}$), the energy-momentum tensor of the chiral species (labeled by $T_{\mu\nu}^{(R)}$), the dissipative contribution $T_{\mu\nu}^{(diss)}$ and the gauge contribution $\mathcal{T}_{\mu\nu}$ (corresponding to an hypercharge gauge-field strength $Y_{\mu\nu}$):

$$T_{\mu\nu}^{(tot)} = T_{\mu\nu}^{(+)} + T_{\mu\nu}^{(-)} + T_{\mu\nu}^{(R)} + T_{\mu\nu}^{(diss)} + \mathcal{T}_{\mu\nu}. \quad (2.1)$$

The covariant conservation of $T_{\mu\nu}^{(tot)}$ implies, as usual, that $\nabla_\mu T_{(tot)}^{\mu\nu} = 0$ where ∇_μ denotes the covariant derivative² defined from the metric tensor $g_{\mu\nu}$ (with signature mostly minus). The chiral and the conduction currents coexist but are not bound to coincide: they obey different equations. More specifically the anomalous current is not covariantly conserved and its evolution can be written as³

$$\nabla_\mu j_R^\mu = \mathcal{A}_R Y_{\alpha\beta} \tilde{Y}^{\alpha\beta}, \quad j_R^\mu = \tilde{n}_R u_R^\mu + \nu_R^\mu, \quad (2.2)$$

²The discussion will be conducted in a general relativistic formulation even if the spectrum of AMHD will be discussed in flat space-time.

³Note that \tilde{n}_R and u_R^μ are respectively the concentration and the four-velocity of the chiral species.

where \mathcal{A}_R is a numerical factor that is determined by the specific nature of the chiral species and by the coupling to the hypercharge field; note that in the Landau frame $\nu_R^\mu u_\mu^R = 0$. Conversely the conduction current is covariantly conserved, it is a source of the evolution equations of the gauge fields and it may even contain a dissipative contribution:

$$\nabla_\mu Y^{\mu\nu} = 4\pi j^\nu, \quad j^\nu = j_+^\nu + j_-^\nu, \quad \nabla_\mu j^\mu = 0, \quad (2.3)$$

where $j_\pm^\nu = (q_\pm \tilde{n}_\pm u_\pm^\mu + \nu_\pm^\mu)$. The dual field strength $\tilde{Y}^{\mu\nu}$ obeys, as usual, $\nabla_\mu \tilde{Y}^{\mu\nu} = 0$. This is the approach followed in [7] which differs from other more conventional approaches [15, 16] where the anomalous current and the conduction current are identified. In the present approach the anomalous current is not directly the source of the evolution of the hypercharge.

In a general relativistic description $Y_{\mu\nu}$ and its dual account for the evolution of the hypercharge field ⁴ however the gauge field strength can be decomposed into the hyperelectric and hypermagnetic parts denoted, respectively, by \mathcal{E}^μ and \mathcal{B}^μ :

$$Y_{\alpha\beta} = \mathcal{E}_\alpha u_\beta - \mathcal{E}_\beta u_\alpha + E_{\alpha\beta\rho\sigma} u^\rho \mathcal{B}^\sigma, \quad (2.4)$$

where $E_{\alpha\beta\rho\sigma} = \sqrt{-g} \epsilon_{\alpha\beta\rho\sigma}$ and $\epsilon_{\alpha\beta\rho\sigma}$ is the four-dimensional Levi-Civita symbol while g is the determinant of the metric tensor. The total four-velocity of the system follows from

$$(p + \rho) u^\mu u^\nu = \sum_a [p_{(a)} + \rho_{(a)}] u_{(a)}^\mu u_{(a)}^\nu, \quad (2.5)$$

where $w = (\rho + p)$ denotes the total enthalpy; the sum in Eq. (2.5) runs over all the species of the plasma, both charged and chiral. Close to an equilibrium situation the four-velocity of the anomalous species coincides with the bulk velocity of the plasma and, therefore, $u_R^\mu \simeq u^\mu$. The vorticity four vector can then be defined as:

$$\omega^\mu = \tilde{f}^{\mu\alpha} u_\alpha \equiv \frac{1}{2} E^{\mu\alpha\beta\gamma} u_\alpha f_{\beta\gamma}, \quad f_{\beta\gamma} = \nabla_\beta u_\gamma - \nabla_\gamma u_\beta. \quad (2.6)$$

From Eqs. (2.4) and (2.6) it follows that the four-divergences of \mathcal{E}^μ , \mathcal{B}^μ and ω^μ are given by:

$$w \nabla_\mu \omega^\mu = -2\omega^\alpha \partial_\alpha p - 2\tilde{n} \mathcal{E}^\alpha \omega_\alpha, \quad (2.7)$$

$$w \nabla_\mu \mathcal{B}^\mu = 2w Y_{\rho\sigma} \omega^\rho u^\sigma + u_\mu \partial_\alpha p \tilde{Y}^{\mu\alpha} + u_\mu Y_{\alpha\beta} j^\beta \tilde{Y}^{\mu\alpha}, \quad (2.8)$$

$$w \nabla_\mu \mathcal{E}^\mu = w[4\pi j^\alpha u_\alpha - \tilde{Y}^{\mu\rho} \omega_\mu u_\rho] + Y^{\beta\gamma} u_\beta \partial_\gamma p + Y^{\beta\gamma} u_\beta Y_{\gamma\alpha} j^\alpha, \quad (2.9)$$

where $w = p + \rho$ is the enthalpy density of the fluid. Equations (2.7), (2.8) and (2.9) have been obtained in the globally neutral case where $\tilde{n} = \tilde{n}_+ = \tilde{n}_-$ and $q_+ = q = -q_-$ but they can be easily generalized to the case where the plasma is not globally neutral.

⁴As soon as we speak of hyperelectric and hypermagnetic fields we are implicitly assuming that the plasma has a finite conduction current so that a preferred frame can be selected where the electric fields are suppressed. Even if the electric and magnetic fields are non-relativistic concepts, it is practical to introduce the electric and the magnetic components of the gauge field strength in a generally covariant language.

2.2 First and second principles of thermodynamics

Denoting with μ_R the chemical potential associated with the anomalous species, the first principle of thermodynamics demands:

$$dE = TdS - pdV + \mu_R dN_R, \quad w = \rho + p = T\varsigma + \mu_R \tilde{n}_R. \quad (2.10)$$

The fundamental identity $E = TS - pV + \mu_R N_R$ can be divided by a fiducial volume and the result is the one reported in the second relation of Eq. (2.10) where ς is the entropy density and ρ the total energy density of the system. Combining the two relations of Eq. (2.10) further thermodynamic relations can be obtained⁵. Since the anomaly-induced currents are protected by topology they are not associated with dissipative effects. Thus the entropy production of the plasma must only come, in the relativistic case, from the viscosity coefficients or from the Ohmic contributions but neither from the chiral currents nor from the corresponding diffusive contribution. The absence of dissipative contributions stemming from the anomalous sector demands that the total entropy four-vector must be supplemented by two further coefficients \mathcal{S}_ω and \mathcal{S}_B :

$$\varsigma^\mu = \varsigma u^\mu - \bar{\mu}_R \nu_R^\mu + \mathcal{S}_\omega \omega^\mu + \mathcal{S}_B \mathcal{B}^\mu. \quad (2.11)$$

The covariant conservation of the total energy momentum tensor $T_{\mu\nu}^{(tot)}$ can be written as

$$\nabla_\mu \varsigma^\mu - \frac{\sigma}{T} Y^{\alpha\beta} Y_{\nu\alpha} u^\nu u_\beta - \frac{T_{(diss)}^{\mu\nu}}{T} \nabla_\mu u_\nu = \mathcal{Z}, \quad (2.12)$$

where we assumed, for the sake of simplicity, a global charge neutrality of the plasma and a corresponding Ohmic form for the charged species, namely $\mathcal{P}_\mu^\alpha j^\mu = \sigma Y^{\alpha\nu} u_\nu$ where $\mathcal{P}_\mu^\alpha = \delta_\mu^\alpha - u_\mu u^\alpha$ is the standard projector. The function \mathcal{Z} appearing in Eq. (2.12) is given by

$$\mathcal{Z} = \nabla_\mu \left(\mathcal{S}_\omega \omega^\mu + \mathcal{S}_B \mathcal{B}^\mu \right) - \frac{\nu^\alpha u^\beta}{T} Y_{\alpha\beta} - \partial_\beta \bar{\mu}_R \nu_R^\beta - \mathcal{A}_R \bar{\mu}_R Y_{\alpha\beta} \tilde{Y}^{\alpha\beta}. \quad (2.13)$$

We remark that the specific definition of the entropy four-vector depends on the chemical potential of the system. However, since the coefficient \mathcal{A}_R does not have a definite sign, the anomalous currents may even lead to violation of the second principle of thermodynamics unless \mathcal{Z} vanishes identically.

2.3 Magnetic and vortical coefficients

The vortical and the magnetic currents modify also the diffusive contributions denoted, respectively, by ν^α and ν_R^α in Eq. (2.13). Four different coefficients parametrize the relation between $(\nu^\alpha, \nu_R^\alpha)$ and $(\omega^\alpha, \mathcal{B}^\alpha)$:

$$\nu^\alpha = \Lambda_\omega \omega^\alpha + \Lambda_B \mathcal{B}^\alpha, \quad \nu_R^\alpha = \Lambda_{R\omega} \omega^\alpha + \Lambda_{RB} \mathcal{B}^\alpha, \quad (2.14)$$

⁵Like, for instance, $\varsigma \partial_\alpha T + \tilde{n}_R \partial_\alpha \mu_R = \partial_\alpha p$ or $\partial_\alpha \rho = T \partial_\alpha \varsigma + \mu_R \partial_\alpha \tilde{n}_R$.

where $(\Lambda_\omega, \Lambda_B)$ and $(\Lambda_{R\omega}, \Lambda_{RB})$ all depend on the chemical potential and on the temperature. Using Eqs. (2.7), (2.8) and (2.9) the condition $\mathcal{Z} = 0$ together with the explicit expression of \mathcal{Z} (see Eq. (2.13)) becomes:

$$\begin{aligned} & \left[2\mathcal{S}_B - \left(\frac{\Lambda_\omega}{T} \right) \right] (\omega^\alpha \mathcal{B}_\alpha) + \left[4\bar{\mu}_R \mathcal{A}_R - \left(\frac{\Lambda_B}{T} \right) \right] (\mathcal{E}^\alpha \mathcal{B}_\alpha) \\ & - \frac{2}{w} \sigma_c \omega^\alpha \mathcal{E}^\beta u^\mu \mathcal{B}^\nu E_{\alpha\beta\mu\nu} \mathcal{S}_\omega + \omega^\alpha \mathcal{P}_\alpha + \mathcal{B}^\alpha \mathcal{Q}_\alpha = 0 \end{aligned} \quad (2.15)$$

where \mathcal{P}_α and \mathcal{Q}_α are two differential operators defined respectively, as:

$$\mathcal{P}_\alpha = \partial_\alpha \mathcal{S}_\omega - \frac{2}{w} \mathcal{S}_\omega \partial_\alpha p - \partial_\alpha \bar{\mu}_R \Lambda_{R\omega}, \quad \mathcal{Q}_\alpha = \partial_\alpha \mathcal{S}_B - \frac{\mathcal{S}_B}{w} \mathcal{S}_\omega \partial_\alpha p - \partial_\alpha \bar{\mu}_R \Lambda_{RB}. \quad (2.16)$$

The results of Eqs. (2.13)–(2.15) follow easily if we recall that, by definition, $u^\alpha \omega_\alpha$, $u^\beta \mathcal{E}_\beta$ and $u^\gamma \mathcal{B}_\gamma$ are all vanishing.

To satisfy the condition expressed by Eq. (2.15) the four-vectors multiplying ω^α and \mathcal{B}^α must vanish together with the coefficients of the terms multiplied by $\omega^\alpha \mathcal{B}_\alpha$ and $\mathcal{E}^\alpha \mathcal{B}_\alpha$. We then arrive at the following conditions:

$$\mathcal{P}_\alpha = 0, \quad \mathcal{Q}_\alpha = 0, \quad \Lambda_B = 4\mu_R \mathcal{A}_R, \quad \Lambda_\omega = 2T \mathcal{S}_B, \quad \mathcal{S}_\omega = 0. \quad (2.17)$$

If, as established, $\mathcal{S}_\omega = 0$ then Eq. (2.15) also implies that $\Lambda_{R\omega} = 0$. All the coefficients we ought to determine depend on $\bar{\mu}_R$ and on the pressure. Thus the conditions of Eq. (2.17) are equivalent to the following system of equations:

$$\left(\frac{\partial \mathcal{S}_B}{\partial p} - \frac{\mathcal{S}_B}{w} \right) \partial_\alpha p + \left(\frac{\partial \mathcal{S}_B}{\partial \bar{\mu}_R} - \Lambda_{RB} \right) \partial_\alpha \bar{\mu}_R = 0, \quad (2.18)$$

where $\Lambda_\omega = 2T \mathcal{S}_B$ and $\Lambda_B = 4\mathcal{A}_R \bar{\mu}_R T$. Using some standard thermodynamic relations (giving the partial derivatives of the pressure and of the rescaled chemical potential with respect to the temperature) the various kinetic coefficients can be determined, after some algebra:

$$\mathcal{S}_B(\bar{\mu}_R, T) = T a_B(\bar{\mu}_R), \quad \Lambda_{RB} = \frac{\partial}{\partial \bar{\mu}_R} \left[T a_B(\bar{\mu}_R) \right], \quad (2.19)$$

$$\Lambda_\omega(\bar{\mu}_R, T) = 2 T^2 a_B(\bar{\mu}_R), \quad \Lambda_B(\bar{\mu}_R, T) = 4 \mathcal{A}_R \bar{\mu}_R T, \quad (2.20)$$

where $a_B(\bar{\mu}_R)$ is an arbitrary function of the rescaled chemical potential. Note also that Λ_B is fully determined in terms of the coefficient of the anomaly and it is, in practice, only function of the chemical potential itself since, by definition, $\bar{\mu}_R T = \mu_R$.

In summary, in a globally neutral plasma with an anomalous current, the relativistic second law implies that the non-anomalous current must contain magnetic and vortical contributions. If the plasma is not hypercharge neutral the form of the kinetic coefficients is subjected to a higher degree of arbitrariness since a second chemical potential must be introduced in the analysis (see the appendix of Ref. [7]).

3 High-frequency propagation

3.1 Two-fluid AMHD equations

The kinetic coefficients of Eqs. (2.19) and (2.20) can be redefined, for practical reasons, as:

$$\begin{aligned} c_\omega(\bar{\mu}_R, T) &= 8\pi T^2 a_B(\bar{\mu}_R), & c_B(\bar{\mu}_R, T) &= 16\pi \mathcal{A}_R \bar{\mu}_R T, \\ c_{RB}(\bar{\mu}_R, T) &= \frac{\partial}{\partial \bar{\mu}_R} [T a_B(\bar{\mu}_R)], & c_{R\omega}(\bar{\mu}_R, T) &= 0. \end{aligned} \quad (3.1)$$

The hypermagnetic and hyperelectric fields denoted, in what follows, by \vec{B} and \vec{E} obey the following set of equations

$$\vec{\nabla} \cdot \vec{E} = 4\pi q(\tilde{n}_+ - \tilde{n}_-), \quad \vec{\nabla} \times \vec{E} = -\partial_t \vec{B}, \quad (3.2)$$

$$\vec{\nabla} \times \vec{B} = 4\pi q(\tilde{n}_+ \vec{v}_+ - \tilde{n}_- \vec{v}_-) + c_\omega \vec{\omega} - c_B \vec{B} + \partial_t \vec{E}, \quad (3.3)$$

where \vec{B} is divergenceless (i.e. $\vec{\nabla} \cdot \vec{B} = 0$). The coefficients $c_\omega(\bar{\mu}_R, T)$ and $c_B(\bar{\mu}_R, T)$ multiply, respectively, the vortical and the magnetic currents of Eq. (3.3). The three-vector where $\vec{\omega}$ defines the total vorticity $\vec{\omega} = (\tilde{\rho}_+ \vec{\omega}_+ + \tilde{\rho}_- \vec{\omega}_-) / (\tilde{\rho}_+ + \tilde{\rho}_-)$ and should not be confused with the frequency (denoted by Ω in what follows). The energy densities of the charged species are denoted by $\tilde{\rho}_\pm$. To establish a direct connection with the spectrum of conventional plasmas in the limit of vanishing kinetic coefficients we shall preferentially consider the situation where the charged species are massive, namely $\tilde{\rho}_\pm = \tilde{n}_\pm(m_\pm + 3T_\pm/2)$ and $p_\pm = \tilde{n}_\pm T_\pm$ with $T_\pm/m_\pm \ll 1$.

The remaining coefficients $c_{R\omega}(\bar{\mu}_R, T)$ and $c_{RB}(\bar{\mu}_R, T)$ of Eq. (3.1) affect directly the evolution of the concentrations:

$$\partial_t \tilde{n}_+ + \vec{\nabla} \cdot (\tilde{n}_+ \vec{v}_+) + \frac{1}{q} \vec{\nabla} \cdot (c_\omega \lambda_+ \vec{\omega}_+) - \frac{1}{q} \vec{\nabla} \cdot (c_B \lambda_+ \vec{B}) = 0, \quad (3.4)$$

$$\partial_t \tilde{n}_- + \vec{\nabla} \cdot (\tilde{n}_- \vec{v}_-) - \frac{1}{q} \vec{\nabla} \cdot (c_\omega \lambda_- \vec{\omega}_-) + \frac{1}{q} \vec{\nabla} \cdot (c_B \lambda_- \vec{B}) = 0, \quad (3.5)$$

$$\partial_t \tilde{n}_R + \vec{\nabla} \cdot (\tilde{n}_R \vec{v}_R) + \vec{\nabla} \cdot (c_{R\omega} \vec{\omega}) - \vec{\nabla} \cdot (c_{RB} \vec{B}) = -4\mathcal{A}_R \vec{E} \cdot \vec{B}, \quad (3.6)$$

where $\lambda_\pm = \tilde{\rho}_\pm / (\tilde{\rho}_+ + \tilde{\rho}_-)$. Concerning Eqs. (3.2)–(3.3) and Eqs. (3.4)–(3.6) few comments are in order. If $\tilde{n}_+ \neq \tilde{n}_-$ a second chemical potential μ_Y (corresponding to the hypercharge) can be introduced in Eq. (2.10). The global hypercharge neutrality of the plasma implies $\mu_Y = 0$. The peculiar velocities determining the currents obey the following set of equations:

$$\partial_t \vec{v}_- + (\vec{v}_- \cdot \vec{\nabla}) \vec{v}_- = -q \frac{\tilde{n}_-}{\tilde{\rho}_-} [\vec{E} + \vec{v}_- \times \vec{B}] + \Gamma_c (\vec{v}_+ - \vec{v}_-) - \frac{\vec{\nabla} p_+}{\rho_+}, \quad (3.7)$$

$$\partial_t \vec{v}_+ + (\vec{v}_+ \cdot \vec{\nabla}) \vec{v}_+ = q \frac{\tilde{n}_+}{\tilde{\rho}_+} [\vec{E} + \vec{v}_+ \times \vec{B}] + \Gamma_c \frac{\tilde{\rho}_-}{\tilde{\rho}_+} (\vec{v}_- - \vec{v}_+) - \frac{\vec{\nabla} p_-}{\rho_-}, \quad (3.8)$$

$$\partial_t \vec{v}_R + (\vec{v}_R \cdot \vec{\nabla}) \vec{v}_R = 0, \quad (3.9)$$

where the pressure gradients shall be eventually neglected; Γ_c denotes the collision frequency determining the generalized conductivity in the single fluid limit.

3.2 Linearization of the two-fluid equations

Equations (3.2)–(3.3), (3.4)–(3.6) and (3.7)–(3.9) will now be linearized in the presence of the weak background magnetic field \vec{B}_0 with the aim of deriving the dispersion relations. The background field will be considered homogeneous: this means that the variation of \vec{B}_0 occurs over typical length-scales⁶ much larger than $1/c_B$. The fluctuations of the various quantities will be introduced as follows:

$$\tilde{n}_\pm(t, \vec{x}) = n_0 + \delta\tilde{n}_\pm(t, \vec{x}), \quad \tilde{n}_R(t, \vec{x}) = n_1 + \delta\tilde{n}_R(t, \vec{x}), \quad \vec{B}(t, \vec{x}) = \vec{B}_0 + \delta\vec{B}(t, \vec{x}), \quad (3.10)$$

while for the other quantities (i.e. $\vec{v}_\pm(t, \vec{x}) = \delta\vec{v}_\pm(t, \vec{x})$, $\vec{v}_R(t, \vec{x}) = \delta\vec{v}_R(t, \vec{x})$ and $\vec{E}(t, \vec{x}) = \delta\vec{E}(t, \vec{x})$) the fluctuations coincide with the field itself. In Eq. (3.10) n_0 and n_1 are, respectively, the uniform background charge and the uniform chiral concentration. The homogeneous value of the chemical potential is related to n_1 and the kinetic coefficients will also be homogeneous. In the case of approximate thermal equilibrium the chemical potential can be related to the concentration as $\bar{\mu}_R = \mu_0 \tilde{n}_R / \varsigma$ where ς denotes the entropy density at equilibrium and where μ_0 is a numerical constant. Therefore if \tilde{n}_R is perturbed around a homogeneous background the kinetic coefficients will also be, in the first approximation homogeneous. Thanks to Eq. (3.10) the perturbed version of the evolution of the concentrations can be written as:

$$\delta\tilde{n}'_\pm + n_0(\vec{\nabla} \cdot \delta\vec{v}_\pm) = 0, \quad \delta\tilde{n}'_R + n_1(\vec{\nabla} \cdot \delta\vec{v}_R) = -4\mathcal{A}_R \delta\vec{E} \cdot \vec{B}_0, \quad (3.11)$$

where the prime denotes a derivation with respect to the time coordinate t . Since the kinetic coefficients are homogeneous in the first approximation, their contribution disappears from Eq. (3.11). With the same notations Eqs. (3.7), (3.8) and (3.9) imply instead:

$$\delta\vec{v}'_\pm = \pm \frac{q}{m_\pm} \left[\delta\vec{E} + \delta\vec{v}_\pm \times \vec{B}_0 \right], \quad \delta\vec{v}'_R = 0, \quad (3.12)$$

where Γ_c has been neglected but it will become relevant at low frequencies, as we shall see later. Finally, after inserting Eq. (3.10) into Eqs. (3.2) and (3.3) we obtain:

$$\vec{\nabla} \cdot \delta\vec{E} = 4\pi q(\delta\tilde{n}_+ - \delta\tilde{n}_-), \quad \vec{\nabla} \cdot \delta\vec{B} = 0, \quad \vec{\nabla} \times \delta\vec{E} = -\delta\vec{B}', \quad (3.13)$$

$$\begin{aligned} \vec{\nabla} \times \delta\vec{B} &= \delta\vec{E}' + 4\pi q n_0(\delta\vec{v}_+ - \delta\vec{v}_-) \\ &- c_B \delta\vec{B} + c_\omega \left[\lambda_+ \vec{\nabla} \times \delta\vec{v}_+ + \lambda_- \vec{\nabla} \times \delta\vec{v}_- \right]. \end{aligned} \quad (3.14)$$

From Eqs. (3.12) the equation obeyed by $\delta\vec{\omega}_\pm$ can also be deduced and they are $\delta\vec{\omega}'_\pm = \pm q[-\delta\vec{B}' + \vec{\nabla} \times (\delta\vec{v}_\pm \times \vec{B}_0)]/m_\pm$. Recalling the standard vector identities⁷ the equation for

⁶ In section 4 we shall specifically discuss also the opposite limit where \vec{B}_0 varies appreciably over typical lengths $L < 1/c_B$ and we shall see that, in this case, the background solution belongs to the class of Beltrami fields.

⁷In particular we recall that $\vec{\nabla} \times (\vec{a} \times \vec{b}) = [\vec{a}(\vec{\nabla} \cdot \vec{b}) - \vec{b}(\vec{\nabla} \cdot \vec{a}) + (\vec{b} \cdot \vec{\nabla})\vec{a} - (\vec{a} \cdot \vec{\nabla})\vec{b}]$.

$\delta\vec{\omega}_\pm$ can also be expressed as:

$$\delta\vec{\omega}'_\pm = \pm \frac{q}{m_\pm} \left[-\delta\vec{B}' - \vec{B}_0(\vec{\nabla} \cdot \delta\vec{v}_\pm) + (\vec{B}_0 \cdot \vec{\nabla})\delta\vec{v}_\pm \right]. \quad (3.15)$$

From Eqs. (3.12) and (3.14) the relevant dispersion relations and the associated refraction indices can be obtained by treating separately the motions parallel and perpendicular to the magnetic field direction.

3.3 Appleton-Hartree determinant

While in conventional plasmas the Appleton-Hartree dispersion relation has been extensively discussed in the literature [6, 17], the AMHD equations linearized in the two-fluid limit contain vortical and magnetic currents. The Laplace transform of Eq. (3.14) implies the following equation

$$(\vec{\nabla} \times \delta\vec{B})_\Omega = -i\Omega\epsilon_s(\Omega) \cdot \delta\vec{E}_\Omega - c_B \delta\vec{B}_\Omega + i c_\omega \vec{\nabla} \times [\epsilon_v(\Omega) \cdot \delta\vec{E}_\Omega], \quad (3.16)$$

where Ω is the frequency (not to be confused with the total vorticity). In Eq. (3.16) $\epsilon_s(\Omega)$ and $\epsilon_v(\Omega)$ denote, respectively, the standard and the vortical components of the dielectric tensor. The explicit form of $\epsilon_s(\Omega)$ and $\epsilon_v(\Omega)$ can be found in appendix A; taking then the curl of Eq. (3.13) and using Eq. (3.16) we obtain the following equation:

$$\vec{\nabla} \times (\vec{\nabla} \times \delta\vec{E}_\Omega) = \Omega^2 \epsilon_s(\Omega) \cdot \delta\vec{E}_\Omega - c_B \vec{\nabla} \times \delta\vec{E}_\Omega - \Omega c_\omega \vec{\nabla} \times [\epsilon_v(\Omega) \cdot \delta\vec{E}_\Omega]. \quad (3.17)$$

We can now go to Fourier space and write Eq. (3.17) as:

$$-\vec{k} \times \vec{k} \times \delta\vec{E}_{\vec{k}\Omega} = \Omega^2 \epsilon_s(\Omega) \cdot \delta\vec{E}_{\vec{k}\Omega} - i c_B \vec{k} \times \delta\vec{E}_{\vec{k}\Omega} - i c_\omega \Omega \vec{k} \times [\epsilon_v(\Omega) \cdot \delta\vec{E}_{\vec{k}\Omega}]. \quad (3.18)$$

We can therefore introduce the refractive index⁸ n satisfying $n = k/\Omega$ where $k = |\vec{k}|$; choosing the coordinate system as $\vec{k} = (0, n\Omega \sin \theta, n\Omega \cos \theta)$ we can obtain from Eq. (3.18) the following Appleton-Hartree matrix:

$$\begin{pmatrix} [1 - \frac{\epsilon_1}{n^2} + c_\omega \frac{\epsilon_4}{n} c(\theta)] & -i[\frac{\epsilon_2}{n^2} + \frac{c_B}{n\Omega} c(\theta) + \frac{c_\omega}{n} \epsilon_3 c(\theta)] & i \frac{c_B}{n\Omega} s(\theta) \\ i[\frac{\epsilon_2}{n^2} + \frac{c_B}{n\Omega} c(\theta) + \frac{c_\omega}{n} \epsilon_3 c(\theta)] & [c^2(\theta) - \frac{\epsilon_1}{n^2} + \frac{\epsilon_4(\Omega)}{n} c_\omega c(\theta)] & -s(\theta) c(\theta) \\ -i \frac{c_B}{n\Omega} s(\theta) - i \frac{c_\omega}{n} \epsilon_3(\Omega) & -s(\theta) c(\theta) + \frac{c_\omega}{n} \epsilon_4 s(\theta) & [s^2(\theta) - \frac{\epsilon_{||}(\omega)}{n^2}] \end{pmatrix}. \quad (3.19)$$

The above matrix reduces to the standard form of the Appleton-Hartree matrix in the limit $c_\omega \rightarrow 0$ and $c_B \rightarrow 0$.

⁸The refractive index cannot be confused with the concentrations denoted by \tilde{n}_\pm and \tilde{n}_R ; their homogeneous values n_0 and n_1 carry specific subscripts so that the notations are clearly established.

4 Dispersion relations

The determinant of the Appleton-Hartree matrix obtained in Eq. (3.19) leads to the following expression:

$$\begin{aligned} & \sin^2 \theta (\varepsilon_{\parallel} - n^2) [n^2 (\varepsilon_L + \varepsilon_R) - 2\varepsilon_R \varepsilon_L] - 2 \cos^2 \theta \varepsilon_{\parallel} (n^2 - \varepsilon_L)(n^2 - \varepsilon_R) \\ & + 2n^6 [c_B^2 f_B(\varepsilon, \Omega, n, \theta) + c_B g_B(\varepsilon, \Omega, n, \theta) + c_{\omega}^2 f_{\omega}(\varepsilon, \Omega, n, \theta) + c_{\omega} g_{\omega}(\varepsilon, \Omega, n, \theta) \\ & c_B^2 c_{\omega} h_1(\varepsilon, \Omega, n, \theta) + c_B c_{\omega} h_2(\varepsilon, \Omega, n, \theta) + c_{\omega}^2 c_B h_3(\varepsilon, \Omega, n, \theta)] = 0. \end{aligned} \quad (4.1)$$

Equation (4.1) is written in terms of the 7 functions explicitly reported in Eq. (A.7) of appendix A. These functions have a specific dependence upon the dielectric tensors; with a collective notation such a dependence has been indicated by ε . The notations followed in Eq. (4.1) imply that c_B^2 multiplies f_B , c_{ω}^2 multiplies f_{ω} ; g_B and g_{ω} multiply, respectively, c_B and c_{ω} ; the three functions h_1 , h_2 and h_3 multiply instead the mixed products. Finally both in Eqs. (4.1) and in Eq. (A.7) we have introduced $\varepsilon_L = (\varepsilon_1 + \varepsilon_2)$ and $\varepsilon_R = (\varepsilon_1 - \varepsilon_2)$ given by:

$$\varepsilon_L(\Omega) = 1 - \frac{\Omega_p^2}{(\Omega + \Omega_{B-})(\Omega - \Omega_{B+})}, \quad \varepsilon_R(\Omega) = 1 - \frac{\Omega_p^2}{(\Omega + \Omega_{B+})(\Omega - \Omega_{B-})}, \quad (4.2)$$

where $\Omega_p^2 = (\Omega_{p+}^2 + \Omega_{p-}^2)$. When $c_B = c_{\omega} = 0$ the magnetic and the vortical currents are absent from the two-fluid AMHD equations and Eq. (4.1) implies the standard result [6]:

$$\sin^2 \theta \left(\frac{1}{n^2} - \frac{1}{\varepsilon_{\parallel}} \right) \left[\frac{1}{2} \left(\frac{1}{\varepsilon_L} + \frac{1}{\varepsilon_R} \right) - \frac{1}{n^2} \right] = \cos^2 \theta \left(\frac{1}{\varepsilon_L} - \frac{1}{n^2} \right) \left(\frac{1}{\varepsilon_R} - \frac{1}{n^2} \right). \quad (4.3)$$

The dispersion relations for a wave propagating parallel (i.e. $\theta = 0$) and perpendicular (i.e. $\theta = \pi/2$) to the magnetic field direction can be obtained easily derived from Eq. (4.3). If $\theta = 0$ Eq. (4.3) reduces to $(n^2 - \varepsilon_R)(n^2 - \varepsilon_L) = 0$ while for $\theta = \pi/2$ Eq. (4.3) implies $(n^2 - \varepsilon_{\parallel})[n^2(\varepsilon_L + \varepsilon_R) - 2\varepsilon_L \varepsilon_R] = 0$. These dispersion relations give therefore the conventional results⁹ which will be generalized hereunder.

4.1 Free-field propagation

In the absence of magnetic field there is no preferred direction and the dispersion relations follow from Eqs. (4.3) and (A.7) by setting all the Larmor frequencies to zero. In this case $\varepsilon_R = \varepsilon_L = \varepsilon_{\parallel}$ and the dispersion relations stem from the following two conditions, namely:

$$\varepsilon_{\parallel}(\Omega) = 0, \quad (n^2 - \varepsilon_{\parallel})\Omega \mp n c_B = 0. \quad (4.4)$$

⁹Along $\theta = 0$ we thus obtain usual dispersion relations for the two circular polarizations of the electromagnetic wave, i.e. $n^2 = \varepsilon_R$ and $n^2 = \varepsilon_L$, while along $\theta = \pi/2$ we have the dispersion relations for the “ordinary” (i.e. $n^2 = \varepsilon_{\parallel}$) and “extraordinary” (i.e. $n^2 = 2\varepsilon_R \varepsilon_L / (\varepsilon_R + \varepsilon_L)$) plasma waves.

Equation (4.4) demonstrates that the vortical current does not contribute to the dispersion relations in the free-field case: c_ω is absent from Eq. (4.4) since the two-fluid effects cancel in the total vorticity. This cancellation is either exact (as in the case of the free-field propagation) or approximate (as we shall see later in the presence of the magnetic field). Indeed, as it can be explicitly verified from Eqs. (A.1), (A.5) and (A.6), $\epsilon_v(\Omega) \rightarrow 0$ when $B_0 \rightarrow 0$: in the limit $B_0 \rightarrow 0$ the vorticity of positively and negatively charged species is balanced so that the net total vorticity vanishes.

The dispersion relation $\varepsilon_{\parallel}(\Omega) = 0$ implies $\Omega = \Omega_p$. This wave does not propagate since its group velocity vanishes and these are nothing but the electrostatic plasma oscillations [6]. The solution of the second equation in Eq. (4.4) is instead¹⁰

$$n = \pm \frac{c_B}{2\Omega} + \sqrt{1 - \frac{\Omega_p^2}{\Omega^2} + \frac{c_B^2}{4\Omega^2}}. \quad (4.5)$$

Equation (4.5) implies also $\Omega^2 = \Omega_p^2 + k^2 \mp kc_B$; these modes are propagating but only affected by the magnetic current, as previously remarked. The birefringent nature of the dispersion relations will be discussed a bit later since this free-field effect may interfere with the presence of the background magnetic field.

If $c_B \rightarrow 0$ and $c_\omega \rightarrow 0$ we have that $n \rightarrow 0$ whenever one of the following three possibility are separately verified $\varepsilon_{\parallel}(\Omega) = 0$ or $\varepsilon_L(\Omega) = 0$ or even $\varepsilon_R(\Omega) = 0$. The frequencies arising from the previous conditions are cut-offs because, for given equilibrium conditions, they define frequencies above or below which the wave ceases to propagate at any angle ($k \rightarrow 0$ for finite Ω , i.e. $v_p = \Omega/k \rightarrow \infty$). This is what happens, in particular, with the dispersion relation of Eq. (4.5). Let us finally remark that the remaining two cut-offs stemming from the conditions $\varepsilon_L(\Omega) = 0$ and $\varepsilon_R(\Omega) = 0$ in Eqs. (4.2) are given, respectively, by:

$$\Omega_R = \sqrt{\Omega_p^2 + (\Omega_{B+} + \Omega_{B-})^2/4} - (\Omega_{B+} + \Omega_{B-})/2, \quad (4.6)$$

$$\Omega_L = \sqrt{\Omega_p^2 + (\Omega_{B+} + \Omega_{B-})^2/4} + (\Omega_{B+} + \Omega_{B-})/2. \quad (4.7)$$

4.2 Parallel propagation

Taking the limit $\theta \rightarrow 0$ in Eq. (4.1) and recalling the results of Eq. (A.7) we obtain:

$$\varepsilon_{\parallel} \{n c_B + [n^2 + n c_\omega (\varepsilon_3 + \varepsilon_4) - \varepsilon_R] \Omega\} \{n c_B - [n^2 + n c_\omega (-\varepsilon_3 + \varepsilon_4) - \varepsilon_L] \Omega\} = 0. \quad (4.8)$$

If $\varepsilon_{\parallel}(\Omega) = 0$ we go back to the case of electrostatic oscillations. Therefore, assuming $\varepsilon_{\parallel}(\Omega) \neq 0$, Eq. (4.8) implies that the standard dispersion relations are modified as:

$$n = \frac{1}{2\Omega} \left[-c_B - c_\omega (\varepsilon_3 + \varepsilon_4) \Omega \pm \sqrt{4\varepsilon_R \Omega^2 + [c_B + c_\omega \Omega (\varepsilon_3 + \varepsilon_4)]^2} \right], \quad (4.9)$$

$$n = \frac{1}{2\Omega} \left[c_B + c_\omega (\varepsilon_3 - \varepsilon_4) \Omega \pm \sqrt{4\varepsilon_L \Omega^2 + [c_B + c_\omega \Omega (\varepsilon_3 - \varepsilon_4)]^2} \right]. \quad (4.10)$$

¹⁰The positive square root has been chosen in Eq. (4.5) in order to get $\Omega > 0$; we consider only positive Ω since solutions with $\Omega < 0$ simply correspond to waves travelling in the opposite direction.

Thus the dispersion relations for the generalized L -mode and R -mode are given, respectively, by:

$$\Omega^2 \varepsilon_R(\Omega) = k^2 + k[c_B + c_\omega(\varepsilon_3 + \varepsilon_4)\Omega], \quad (4.11)$$

$$\Omega^2 \varepsilon_L(\Omega) = k^2 - k[c_B + c_\omega(\varepsilon_3 - \varepsilon_4)\Omega]. \quad (4.12)$$

In the high-frequency limit (i.e. formally $\Omega \rightarrow \infty$) we have that $c_\omega(\varepsilon_3 \pm \varepsilon_4)\Omega \rightarrow 0$ since, from Eqs. (A.5)–(A.6), we have:

$$c_\omega(\varepsilon_3 + \varepsilon_4)\Omega = \frac{q c_\omega}{(m_+ + m_-)} \left[\frac{\Omega}{\Omega_{B-} - \Omega} + \frac{\Omega}{\Omega_{B+} + \Omega} \right], \quad (4.13)$$

$$c_\omega(\varepsilon_3 - \varepsilon_4)\Omega = \frac{q c_\omega}{(m_+ + m_-)} \left[\frac{\Omega}{\Omega - \Omega_{B+}} - \frac{\Omega}{\Omega + \Omega_{B-}} \right]. \quad (4.14)$$

The results of Eqs. (4.11)–(4.12) and (4.13)–(4.14) demonstrate, once more, that in the high-frequency limit of the spectrum the magnetic current dominates against the vortical current. For intermediate frequencies (i.e. as soon as we reduce Ω) the terms containing the natural frequencies of the plasma come then into play so that for the R and L modes the corresponding dispersion relations become:

$$\begin{aligned} \Omega^2 &= k^2 + k c_B + \frac{\Omega_p^2 \Omega}{(\Omega - \Omega_{B-})}, & R\text{-mode}, \\ \Omega^2 &= k^2 - k c_B + \frac{\Omega_p^2 \Omega}{(\Omega + \Omega_{B-})}, & L\text{-mode}. \end{aligned} \quad (4.15)$$

As in the standard case, the phase velocity of the R -mode is greater than that of the L -mode. In Eq. (4.15) we assumed $m_+ > m_-$ and therefore $\Omega_{B+} < \Omega_{B-}$. In the limit $k \rightarrow 0$ the R -mode cut-off occurs above Ω_p while the L -mode cut-off occurs below Ω_p (i.e., recalling Eqs. (4.6) and (4.7), $\Omega \rightarrow \Omega_R$ and $\Omega \rightarrow \Omega_L$). In the low-frequency limit ε_R and ε_L coincide to leading order in (Ω/Ω_{B+}) and in (Ω/Ω_{B-}) since

$$\lim_{\Omega \rightarrow 0} \varepsilon_R(\Omega) = \lim_{\Omega \rightarrow 0} \varepsilon_L(\Omega) \rightarrow 1 + \frac{\Omega_p^2}{\Omega_{B+} \Omega_{B-}} = 1 + \frac{1}{v_A^2}, \quad v_A = \frac{B_0}{\sqrt{4\pi n_0(m_+ + m_-)}}, \quad (4.16)$$

where v_A denotes the Alfvén velocity of the system. In the low-frequency limit the dispersion relations for the R -mode and for the L -mode are, respectively,

$$\Omega^2 = \frac{v_A^2}{1 + v_A^2} \left\{ k^2 + k \left[c_B + \frac{q c_\omega}{m} \left(\frac{\Omega}{\Omega_{B+}} + \frac{\Omega}{\Omega_{B-}} \right) \right] \right\}, \quad (4.17)$$

$$\Omega^2 = \frac{v_A^2}{1 + v_A^2} \left\{ k^2 - k \left[c_B - \frac{q c_\omega}{m} \left(\frac{\Omega}{\Omega_{B+}} + \frac{\Omega}{\Omega_{B-}} \right) \right] \right\}, \quad (4.18)$$

since $v_A \ll 1$ we also have $v_A^2/(1 + v_A^2) \simeq v_A^2$.

Having determined the dispersion relations in the case of parallel propagation, the Faraday rotation rate can be easily determined with the standard procedure. The generalized Faraday rotation angle experienced by the linearly polarized radiation travelling parallel to the magnetic field direction can be obtained as

$$\Delta\Phi = \frac{\Omega}{2} \left\{ \frac{c_B}{\Omega} + c_\omega \varepsilon_3 + \sqrt{\varepsilon_L + \left[\frac{c_B}{2\Omega} + \frac{c_\omega}{2}(\varepsilon_3 + \varepsilon_4) \right]^2} - \sqrt{\varepsilon_R + \left[\frac{c_B}{2\Omega} + \frac{c_\omega}{2}(\varepsilon_3 - \varepsilon_4) \right]^2} \right\} \Delta L, \quad (4.19)$$

where ΔL is the distance travelled by the signal in the direction parallel to the magnetic field direction. It is interesting to compare the contribution of the terms depending upon c_B and those depending upon the background magnetic field intensity, i.e. the terms appearing in the squared brackets. Recalling the expressions of $(\varepsilon_R, \varepsilon_L)$ we have that $\Omega_{B+} \ll \Omega_{B-}$ and $\Omega_{p+} \ll \Omega_{p-}$ (always assuming $m_+ \gg m_-$). In this case $\Delta\Phi/\Delta L$ interpolates between the standard result $(\Omega_{B-}/2)(\Omega_{p-}/\Omega)^2$ (valid when $c_B \rightarrow 0$) and the constant rotation rate $c_B/2$ (valid when $B_0 \rightarrow 0$ as in the case of free-field propagation). As it can be explicitly verified the c_ω is subdominant at high frequencies and can be neglected.

4.3 Orthogonal propagation

By setting $\theta \rightarrow \pi/2$ in Eq. (4.1) and recalling the results of Eq. (A.7) we obtain the following simple equation:

$$n^4(\varepsilon_L + \varepsilon_R)\Omega^2 - n^2\{c_B^2(\varepsilon_L + \varepsilon_R) + c_B c_\omega[\varepsilon_4(-\varepsilon_L + \varepsilon_R) + \varepsilon_3(\varepsilon_L + \varepsilon_R)]\Omega + [\varepsilon_\parallel \varepsilon_R + \varepsilon_L(\varepsilon_\parallel + 2\varepsilon_R)]\Omega^2\} + 2\varepsilon_L \varepsilon_R \Omega^2 \varepsilon_\parallel = 0. \quad (4.20)$$

The solution of Eq. (4.20) can be obtained by first solving in terms of n^2 . The result is

$$n^2 = \frac{\mathcal{J}(\varepsilon, \Omega) \pm \sqrt{\mathcal{M}(\varepsilon, \Omega)}}{2(\varepsilon_R + \varepsilon_L)\Omega^2}, \quad (4.21)$$

$$\begin{aligned} \mathcal{J}(\varepsilon, \Omega) &= c_B^2(\varepsilon_L + \varepsilon_R) + c_B c_\omega[\varepsilon_4(-\varepsilon_L + \varepsilon_R) + \varepsilon_3(\varepsilon_L + \varepsilon_R)]\Omega \\ &+ [\varepsilon_\parallel \varepsilon_R + \varepsilon_L(\varepsilon_\parallel + 2\varepsilon_R)]\Omega^2, \end{aligned} \quad (4.22)$$

$$\begin{aligned} \mathcal{M}(\varepsilon, \Omega) &= -8\varepsilon_\parallel \varepsilon_R \varepsilon_L(\varepsilon_L + \varepsilon_R)\Omega^4 + \{c_B^2(\varepsilon_L + \varepsilon_R) \\ &+ c_B c_\omega[\varepsilon_4(\varepsilon_R - \varepsilon_L) + \varepsilon_3(\varepsilon_L + \varepsilon_R)]\Omega + [\varepsilon_\parallel \varepsilon_R + \varepsilon_L(\varepsilon_\parallel + 2\varepsilon_R)]\Omega^2\}^2. \end{aligned} \quad (4.23)$$

Equation (4.21) in the limit $c_\omega \rightarrow 0$ and $c_B \rightarrow 0$ reduces to the ordinary mode if we choose the plus (i.e. $n^2 = \varepsilon_\parallel$) and to the extraordinary mode (i.e. $n^2 = 2\varepsilon_R \varepsilon_L/(\varepsilon_R + \varepsilon_L)$) if we choose the minus. In the high-frequency limit the terms multiplying the vortical current are always negligible as already remarked in the case of the parallel propagation. The phenomena related to the oblique propagation will not be specifically discussed.

4.4 Spectrum around a hypermagnetic knot

Introducing the three mutually orthogonal unit vectors $\hat{a}(z, p)$, $\hat{b}(z, p)$ and \hat{z} defined in appendix B, we can consider the modes of fluctuation of the hypermagnetic field around a fully inhomogeneous background $\vec{B}_0(t, \vec{x})$, namely:

$$\vec{B}(t, \vec{x}) = \vec{B}_0(t, \vec{x}) + \delta\vec{B}(t, \vec{x}). \quad (4.24)$$

Since the background solution is not uniform we can align \vec{B}_0 along $\hat{a}(z, p)$ and write that $\vec{B}_0(z) = B_0 \hat{a}(z, p)$. The background equations are solved by setting $p = -c_B$ (since $\vec{\nabla} \times \vec{B}_0 = p \vec{B}_0$). As in the homogeneous case the velocities vanish on the background solution. For $L < 1/c_B$ the background field is homogeneous and the previous analyses apply. For typical length-scales larger than the scale of spatial variation of hypermagnetic knot (i.e. $L \gg 1/c_B$) there are two separate possibilities for the perturbed velocity field: either $\delta\vec{v} \parallel \vec{B}_0$ or $\delta\vec{v} \perp \vec{B}_0$. These two cases will now be separately examined.

The case of parallel propagation mirrors exactly the one already discussed in the case of uniform field. If we assume that $\delta\vec{v} \parallel \vec{B}_0$ the dispersion relations follow from

$$\{[k^2 - \Omega^2 \varepsilon_{\parallel}(\Omega)]^2 - c_B^2 k^2\} \varepsilon_{\parallel}(\Omega) = 0. \quad (4.25)$$

The parallel dielectric tensor is $\varepsilon_{\parallel}(\Omega) = 1 - \Omega_p^2/[\Omega(\Omega + i\Gamma_c)]$ where the correction coming from the collision rate has been added for immediate convenience. The solution $\varepsilon_{\parallel}(\Omega) = 0$ gives, as before, the electrostatic wave. The solution of $\{[k^2 - \Omega^2 \varepsilon_{\parallel}(\Omega)]^2 - c_B^2 k^2\} = 0$ gives, respectively, a high-frequency and a low-frequency branch. The high-frequency branch has the same dispersion relation of the free-field case, namely $\Omega^2 \simeq \Omega_p^2 + k^2 \mp kc_B$. The low-frequency branch is instead derived from the explicit form of the dispersion relation written as:

$$\Omega + i\Gamma_c = \frac{\Omega^2}{k^2}(\Omega + i\Gamma_c) - \frac{\Omega_p^2 \Omega}{k^2} \pm \frac{c_B}{k}(\Omega + i\Gamma_c); \quad (4.26)$$

neglecting the first term at the right-hand side of the previous equation (which is unimportant at low frequencies) we have that:

$$\Omega = -\frac{i\Gamma_c(1 \mp c_B/k)}{1 + \Omega_p^2/k^2 \mp c_B/k}. \quad (4.27)$$

The low-frequency mode, in which the conducting current dominates over the displacement current, has no counterpart in vacuum. In the low-frequency mode, a small electric field proportional to Γ_c exist to give the necessary current parallel to the magnetic field. In the limit $\Gamma_c \rightarrow 0$ both the electric field and resistivity vanish and the low-frequency mode becomes the force-free field. As expected the same phenomenon occurs in the absence of magnetic and vortical currents [20, 21].

In the case of orthogonal propagation the fluctuations of the hypermagnetic field and of the velocity can be decomposed by using the gyrotropic basis of appendix B:

$$\delta\vec{B}(t, z) = \delta B_1(t) \hat{b}(z, p) + \delta B_2(t) \hat{z}, \quad \delta\vec{v}^{(\pm)}(t, z) = \delta v_1^{(\pm)}(t) \hat{b}(z, p) + \delta v_2^{(\pm)}(t) \hat{z}. \quad (4.28)$$

For a generic velocity fluctuation orthogonal to \vec{B}_0 we have $\delta\vec{v} \times \vec{B}_0 = [B_0(\delta\vec{v} \cdot \hat{z})\hat{b} - B_0(\delta\vec{v} \cdot \hat{b})\hat{z}]$; the solutions for $\delta v_1^{(\pm)}(t)$ and $\delta v_2^{(\pm)}(t)$ can then be expressed as:

$$\begin{aligned}\delta v_1^{(\pm)}(\Omega) &= \frac{q}{m_{\pm}(\Omega_{B\pm}^2 - \Omega^2)} \left[\pm i\Omega \delta E_1 + \Omega_{B\pm} \delta E_2 \right], \\ \delta v_2^{(\pm)}(\Omega) &= \frac{q}{m_{\pm}(\Omega_{B\pm}^2 - \Omega^2)} \left[\pm i\Omega \delta E_2 - \Omega_{B\pm} \delta E_1 \right].\end{aligned}\quad (4.29)$$

The dispersion relations in this case are given by

$$\varepsilon_R \varepsilon_L - \frac{c_B c_\omega}{2\Omega} \left[\varepsilon_L(\varepsilon_3 - \varepsilon_4) + \varepsilon_R(\varepsilon_3 + \varepsilon_4) \right] = 0. \quad (4.30)$$

In the high-frequency limits defined by Eqs. (4.13)–(4.14), Eq. (4.30) is satisfied if $\varepsilon_R \varepsilon_L = 0$ which is verified when either ε_L or ε_R are vanishing. Equation (4.30) leads to vanishing group velocity in the high-frequency regime: the corresponding modes are then not propagating. The proper frequencies defined by these equations have been already derived in Eqs. (4.11) and (4.12).

5 Single fluid description and its implications

The two-fluid equations can now be combined with the purpose of deriving the effective single fluid description valid for sufficiently large length-scales and for frequencies much smaller than Ω_p and $\Omega_{B\pm}$. The one-fluid variables are the total current $\vec{J} = q(n_+, \vec{v}_+ - n_- \vec{v}_-)$, the bulk velocity of the plasma $\vec{v} = (m_+ \vec{v}_+ + m_- \vec{v}_-)/(m_+ + m_-)$ and the total mass density $\rho_m = (m_+ n_+ + m_- n_-)$. In the globally neutral case \vec{J} and ρ_m become, respectively, $\vec{J} = n_0(\vec{v}_+ - \vec{v}_-)$ and $\rho_m = n_0(m_+ + m_-)$. Summing-up Eq. (3.7) (multiplied by m_+) and Eq. (3.8) (multiplied by m_-) the evolution equation for the bulk velocity of the plasma is

$$\rho_m \left[\partial_t \vec{v} + \vec{v} \cdot \vec{\nabla} \vec{v} \right] = \vec{J} \times \vec{B} - \vec{\nabla} P + \eta \nabla^2 \vec{v}, \quad (5.1)$$

where the shear viscosity contribution, labeled by η , has been added for convenience¹¹. Taking the difference of Eqs. (3.7) and (3.8) (and assuming $m_+ > m_-$) the generalized Ohm's law can be written as:

$$\partial_t \vec{J} + \Gamma_c \vec{J} \simeq \frac{\Omega_{p-}^2}{4\pi} \left(\vec{E} + \vec{v} \times \vec{B} + \frac{\vec{\nabla} p_-}{qn_0} - \frac{\vec{J} \times \vec{B}}{qn_0} \right), \quad (5.2)$$

where global neutrality has been assumed. Note that we have also kept the thermoelectric term (depending on the pressure gradient of the lightest charge carriers) and the Hall term. Since we shall mainly consider the case of homogeneous pressures the thermoelectric term will be neglected; the Hall term is a higher order contribution, as we shall argue.

¹¹If the total pressure does not vanish Eq. (5.1) is modified as follows $\partial_t[w \vec{v}] + (\vec{v} \cdot \vec{\nabla})[w \vec{v}] + \vec{v} \cdot \vec{\nabla} \cdot [w \vec{v}] = -\vec{\nabla} P + \vec{J} \times \vec{B} + \eta \nabla^2 \vec{v}$ where w , as already discussed, is the enthalpy density.

In the globally neutral case the single fluid equations stipulate that \vec{E} , \vec{B} and \vec{J} are all solenoidal (i.e. $\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot \vec{J} = 0$). A fourth possible solenoidal vector is the bulk velocity of the plasma \vec{v} . Indeed, since the evolution of ρ_m and $\rho_q = q(n_+ - n_-)$ is given by¹²

$$\partial_t \rho_m + \vec{\nabla} \cdot (\rho_m \vec{v}) = 0, \quad \partial_t \rho_q + \vec{\nabla} \cdot \vec{J} = 0, \quad (5.3)$$

the incompressible closure $\vec{\nabla} \cdot \vec{v} = 0$ will be adopted; consistently with the incompressible closure ρ_m will be considered homogeneous, at least in the first part of this section. A full discussion of other possible closure (such as the ones conventionally adopted in conventional plasmas) is desirable but beyond the scope of this analysis. The remaining one-fluid equation containing the vortical and the magnetic currents, can be written as:

$$\vec{\nabla} \times \vec{B} - \partial_t \vec{E} = 4\pi \vec{J} + c_\omega \vec{\omega} - c_B \vec{B}, \quad \vec{\omega} = \vec{\nabla} \times \vec{v}. \quad (5.4)$$

Since the one-fluid description involves the lowest branch of the spectrum we can neglect the displacement current that becomes relevant only for the electromagnetic propagation. For the same reason we can neglect the time derivative in Eq. (5.2), i.e. $\partial_t \vec{J} \ll \Omega_p^2 \vec{E}$. Consequently Eqs. (5.2) and (5.4) in the low-frequency branch of the spectrum become

$$\vec{\nabla} \times \vec{B} = 4\pi \vec{J} + c_\omega \vec{\omega} - c_B \vec{B}, \quad \vec{E} = \vec{J}/\sigma - \vec{v} \times \vec{B}. \quad (5.5)$$

Recalling that $\vec{\nabla} \times \vec{E} = -\partial_t \vec{B}$, Eq. (5.5) can be used to obtain an equation that is reminiscent of the magnetic diffusivity equation, namely

$$\partial_t \vec{B} = \vec{\nabla} \times (\vec{v} \times \vec{B}) + \frac{\nabla^2 \vec{B}}{4\pi\sigma} + \frac{1}{4\pi\sigma} \vec{\nabla} \times (c_\omega \vec{\omega}) - \frac{1}{4\pi\sigma} \vec{\nabla} \times (c_B \vec{B}). \quad (5.6)$$

Introducing the vorticity $\vec{\omega}$ into Eq. (5.1) and dividing both sides of the equation by ρ_m we obtain

$$\partial_t \vec{v} + \vec{\omega} \times \vec{v} = \frac{\vec{J} \times \vec{B}}{\rho_m} - \vec{\nabla} \left[\frac{P}{\rho_m} + \frac{v^2}{2} \right] + \nu_{kin} \nabla^2 \vec{v}. \quad (5.7)$$

Taking the curl of Eq. (5.7) the evolution equation of the vorticity becomes:

$$\partial_t \vec{\omega} = \vec{\nabla} \times (\vec{v} \times \vec{\omega}) + \frac{\vec{\nabla} \times (\vec{J} \times \vec{B})}{\rho_m} + \nu_{kin} \nabla^2 \vec{\omega}. \quad (5.8)$$

The most interesting solutions of the one-fluid equations will involve the situations where the vortical and the magnetic currents play the dominant role. However, before turning the attention on these classes solutions it is useful to remark that the equilibrium solutions of the plasma at rest (i.e. $\vec{v} = 0$) are simply given by

$$\vec{\nabla} P = \vec{J} \times \vec{B}, \quad 4\pi \vec{J} = \left(\vec{\nabla} \times \vec{B} - c_\omega \vec{\omega} + c_B \vec{B} \right). \quad (5.9)$$

¹² Note that the global neutrality implies $\rho_q = 0$ and Eq. (5.3) demands $\vec{\nabla} \cdot \vec{J} = 0$ in full agreement with the solenoidal nature of the total Ohmic current.

From Eq. (5.9) it is immediate to obtain the following three identities

$$4\pi\vec{\nabla}P = (\vec{\nabla} \times \vec{B}) \times \vec{B}, \quad (\vec{B} \cdot \vec{\nabla})P = 0, \quad (\vec{J} \cdot \vec{\nabla})P = 0. \quad (5.10)$$

The relations of Eq. (5.10) are not explicitly modified by the presence of vortical and magnetic currents. The two last conditions in Eq. (5.10) define the so-called magnetic surfaces: the pressure gradient vanishes along the lines of magnetic force and along the current lines. We conclude that neither the vortical nor the magnetic current affect directly the equilibrium solutions.

5.1 Bulk velocity parallel to the magnetic field direction

The single fluid AMHD equations admit various solutions that have no counterpart in the case of ordinary MHD. Consider first the situation where the hypermagnetic magnetic field and the velocity are parallel and have non-vanishing magnetic and kinetic gyrotropy, i.e.

$$\vec{v} \times \vec{B} = 0, \quad \vec{v} \cdot \vec{\nabla} \times \vec{v} = p_v(t)v^2, \quad \vec{B} \cdot \vec{\nabla} \times \vec{B} = p_B(t)B^2. \quad (5.11)$$

The simplest way to realize the situation described by Eq. (5.11) is to require that \vec{v} and \vec{B} are both Beltrami-like fields (see appendix B for this terminology) characterized by $\vec{\nabla} \times \vec{v} = p_v(t)\vec{v}$ and $\vec{\nabla} \times \vec{B} = p_B(t)\vec{B}$. Moreover, since $\vec{v} \times \vec{B} = 0$, it is natural to require that $p_v(t) = p_B(t)$. From Eq. (5.5) the total current \vec{J} can be easily determined; the Ohmic electric field is then given by:

$$\vec{E} = \frac{p_B(t) + c_B(t)}{4\pi\sigma} \vec{B} - \frac{c_\omega(t)}{4\pi\sigma} \vec{\omega}. \quad (5.12)$$

From Eq. (5.6) the hypermagnetic field is obtained by solving the following equation

$$\partial_t \vec{B} = -\frac{p_B(t)[p_B(t) + c_B(t)]}{4\pi\sigma(t)} \vec{B} + \frac{c_\omega(t)}{4\pi\sigma(t)} p_B(t) \vec{\omega}, \quad (5.13)$$

where $\vec{\omega}(t, z)$ is the solution of Eq. (5.8). Thanks to the symmetries of the problem the solution of this equation is given by $\vec{\omega}(t, z) = \vec{\omega}_0(z) \exp[-\int_0^t p_B^2(t') \nu_{kin}(t') dt']$, where $\omega_0(z)$ is the initial vorticity which can also be written as $\vec{\omega}_0(z) = p_B(0)\vec{v}_0(z)$. Equation (5.13) can then be solved in general terms. However, recalling that $c_B(t)$ and $c_\omega(t)$ are explicit functions of time but they depend on the rescaled chemical potential¹³ since $p_B(t)$ is arbitrary we can choose $p_B(t) = -c_B(t)$. In this case, the solution of Eq. (5.13) shares the same properties of the general solution but it is mathematically simpler:

$$\vec{B}(t, z) = \vec{\omega}_0(z) \int_0^t dt' \frac{c_B(t')c_\omega(t')}{4\pi\sigma(t')} e^{-\int_0^{t'} c_B^2(t'') \nu_{kin}(t'') dt''}. \quad (5.14)$$

¹³In this discussion we shall keep the time-dependence in the kinetic coefficients even if, strictly speaking, c_B and c_ω may depend on the temperature and of the chemical potential but they are constant in time. However, in curved backgrounds a mild breaking of conformal invariance may induce a time dependence which is, however, not central to the present analysis.

The results of Eq. (5.14) describe the generation of the hypermagnetic field thanks to some initial vortical current. To deepen this question let us assume that c_ω and c_B are both constant. Equation (5.14) can then be solved and the result is

$$\vec{B}(z, t) = -\frac{\vec{v}_0(z) c_\omega}{4\pi \nu_{kin} \sigma} \left[1 - e^{-\nu_{kin} c_B^2 t} \right], \quad (5.15)$$

where we used that $\vec{\omega}_0(z) = -c_B \vec{v}_0(z)$ when c_B is constant in time and $c_B(0) = c_B$. This result is also valid for a relativistic equation of state (i.e. $w = 4\rho/3$) provided the incompressible closure is consistently adopted and can be easily generalized to curved backgrounds. While these generalizations are not germane to our theme it is worth to emphasize that in the limit $t \rightarrow \infty$ the suppression of the magnetic field is controlled by $4\pi\sigma\nu_{kin}$ which is nothing but the Prandtl number given as the ratio of the magnetic and of the kinetic Reynolds number [2]. The Prandtl number is roughly independent on the temperature. For instance at the electroweak epoch [10] we would have that $\nu_{kin} \simeq 1/(\alpha'^2 T)$ while $\sigma \simeq T/\alpha'$ where $\alpha' = g'^2/(4\pi)$. Recalling the results of Eqs. (2.19) and (2.20) we therefore have that

$$\lim_{t \rightarrow \infty} \vec{B}(t, z) = \frac{c_\omega(T)}{4\pi} \alpha'^3 \vec{v}_0, \quad c_\omega(T) = 2T^2 a(\bar{\mu}_R), \quad (5.16)$$

where $a_B(\bar{\mu}_R)$ is the usual arbitrary function of the rescaled chemical potential $\bar{\mu}_R = \mu_R/T$ (see Eqs. (2.19) and (2.20)). The suppression due to the conductivity is therefore eliminated and what is left is a milder suppression $\mathcal{O}(\alpha'^3)$. In the same limit the hypermagnetic current turns out to be more suppressed than the vortical current. As long as the inverse of the Prandtl number scales as α'^3 the previous discussion is generally valid and this is what happens in the case of the electroweak plasma [10] (see also [18] for specific estimates of the conductivity in the electroweak phase¹⁴).

5.2 Bulk velocity orthogonal to the magnetic field direction

If the hypermagnetic field and the bulk velocity of the plasma are orthogonal (i.e. $\vec{v} \cdot \vec{B} = 0$), employing the helical basis of appendix B the hypermagnetic field and the velocity field can be written as:

$$\vec{B}(t, z) = B_0(t) \hat{z} + B_1(t) \hat{a}(z, p), \quad \vec{v}(z, t) = v(t) \hat{b}(z, p). \quad (5.17)$$

From Eq. (5.5) the AMHD current becomes:

$$\vec{J}(t, z) = \frac{B_1(t)}{4\pi} [c_B + p] \hat{a}(z, p) - \frac{c_\omega}{4\pi} p v(t) \hat{b}(z, p) + \frac{c_B B_0(t)}{4\pi} \hat{z}. \quad (5.18)$$

By analyzing the structure of the evolution equation of the magnetic field and of the vorticity it emerges that the system is consistent provided $\partial_t B_0 = 0$ (i.e. constant magnetic field along

¹⁴Similar kinds of considerations can also be developed in the case of a strongly interacting plasma as long as the same scaling occurs [19].

\hat{z}) and provided $c_\omega = 0$. In this case the coupled evolution of the vorticity and of the magnetic field obeys

$$\frac{d\omega}{dt} = \frac{pc_B B_0}{4\pi\rho_m} B_1 - \frac{B_0}{4\pi\rho_m} p[c_B + p]B_1, \quad \frac{dB_1}{dt} = \omega B_0 - \frac{pB_1}{4\pi\sigma}[c_B + p], \quad (5.19)$$

where $\omega(t) = p v(t)$ and $\omega(z, t) = \omega(t)\hat{b}(z, p)$. The equations can be diagonalized with a specific choice of the coordinate system. The simplest and most convenient one is $p = -c_B$; in this case the two equations can be combined by differentiating once Eq. (5.19). The result is

$$\frac{d^2 B_1}{dt^2} + c_B^2 v_A^2 B_1 = 0, \quad v_A = \frac{B_0}{\sqrt{4\pi\rho_m}}, \quad (5.20)$$

where v_A is the Alfvén velocity. This solution has been swiftly presented in Ref. [7] and recently rediscovered in [24]. Equation (5.20) describes the AMHD analog of the non-linear Alfvén wave. The anomalous Alfvén wave has been already discussed in section 3 as a low-frequency limit of the two-fluid equations.

5.3 Fully non-linear evolution and baryogenesis

So far we considered small fluctuations of the chiral concentration around an otherwise homogeneous value denoted by n_1 in section 2. In the opposite case the AMHD equations imply a specific relation between the concentration (or the chemical potential) and the topological properties of the hypermagnetic fields. To illustrate this point we will show that close to equilibrium the chemical potential is determined not only by the magnetic gyrotropy but also by the total vorticity of the plasma. Hypermagnetic field configurations with non-vanishing gyrotropy have been used to model the generation of the baryon or lepton asymmetry [10] (see also [25, 26, 27]). Consider, therefore, the evolution equation of the chemical potential which can be written as

$$\partial_t \bar{\mu}_R + \Gamma \bar{\mu}_R = -\frac{4\mu_0}{\varsigma} \mathcal{A}_R \vec{E} \cdot \vec{B}, \quad (5.21)$$

where Γ is the perturbative rate of the chirality flip processes (in the case of [10] it is determined by the scattering of right electrons with the Higgs and gauge bosons and with the top quarks because of their large Yukawa coupling). In Eq. (5.21) we also used the following general relation $\bar{\mu}_R = \mu_0 n_R / \varsigma$ where μ_0 is a numerical factor depending on the specific features of the plasma while ς is, as usual, the entropy density.

To compute $\bar{\mu}_R$ in the proximity of an equilibrium situation we need to deduce the hyperelectric field. Recalling then Eq. (5.5), the hyperelectric field can be related to the total Ohmic current so that Eq. (5.21) will become

$$\partial_t \bar{\mu}_R + \Gamma \bar{\mu}_R - \frac{8 T^2 a_B(\bar{\mu}_R) \mu_0}{\sigma \varsigma} \mathcal{A}_R \vec{\omega} \cdot \vec{B} + 16 \frac{\mu_0 T \bar{\mu}_R}{\sigma \varsigma} \mathcal{A}_R^2 B^2 = -\frac{\mu_0}{\pi \sigma \varsigma} \vec{B} \cdot \vec{\nabla} \times \vec{B}. \quad (5.22)$$

We shall now choose $a_B(\bar{\mu}_R) = \bar{\mu}_R$; if $a_B(\bar{\mu}_R) \neq \bar{\mu}_R$ in the evolution equation of the concentration we should add a further term proportional to $\vec{\nabla} \cdot [c_{RB}(\bar{\mu}_R)\vec{B}]$. This term vanishes in the case $a_B(\bar{\mu}_R) \propto \bar{\mu}_R$: $c_{RB}(\bar{\mu}_R)$ contains the derivative of $a_B(\bar{\mu}_R)$ with respect to $\bar{\mu}_R$ and it is therefore constant. In more general situations Eq. (5.22) will just contain a supplementary contribution of the type $\vec{B} \cdot \vec{\nabla} \bar{\mu}_R$.

The rescaled chemical potential enters the infinitely conducting limit (see appendix B) and the smallness of the particle asymmetries is the rationale for the minuteness of the rescaled chemical potentials in approximate thermal equilibrium. At equilibrium, we can determine $\bar{\mu}_R$ from Eq. (5.22) and the result is given by:

$$\bar{\mu}_R = -\left(\frac{\mu_0 \mathcal{A}_R}{\pi \varsigma \sigma}\right) \frac{\vec{B} \cdot \vec{\nabla} \times \vec{B}}{[\Gamma + \Gamma_B - \Gamma_\omega]}, \quad (5.23)$$

While Γ is the perturbative chirality flip rate, the other terms can be understood as rates stemming from the hypermagnetic current and from the vortical current and they are

$$\Gamma_B = \frac{16\mu_0}{\varsigma \sigma} \mathcal{A}_R^2 T B^2, \quad \Gamma_\omega = \frac{8T^2 \mu_0}{\pi \sigma \varsigma} \mathcal{A}_R \vec{\omega} \cdot \vec{B}. \quad (5.24)$$

In the case of right electrons (see [10]) $\mathcal{A}_R = -g'^2 y_R^2 / (64\pi^2)$ where g' denotes the gauge coupling and $y_R = -2$ is the hypercharge assignment of the right electrons. In the same situation we have $\bar{\mu}_R = \mu_0 n_R / \varsigma$ and $\mu_0 = 87\pi^2 N_{eff} / 220$, where N_{eff} is the effective number of relativistic degrees of freedom of the system¹⁵. If the plasma is hypercharge neutral the value of the chemical potential can be estimated from the asymmetry in the case where all the standard model charges are in complete thermal equilibrium. If all the asymmetry is attributed to the right electrons (which is, in some sense, the most favourable situation) then $\bar{\mu}_R = (87\pi^2 / 220) N_{eff}(n_R / \varsigma)$ where $N_{eff} = 106.75$. With these specifications Eq. (5.23) becomes

$$\bar{\mu}_R = -\frac{783 \alpha'}{88 \pi \sigma T^3} \frac{\vec{B} \cdot \vec{\nabla} \times \vec{B}}{[\Gamma + \Gamma_B - \Gamma_\omega]}, \quad \Gamma_B = \frac{783 \alpha'^2 B^2}{22 \pi^2 \sigma T^3}, \quad (5.25)$$

where $\alpha' = g'^2 / (4\pi)$. Equation (5.25) coincides with the previous results (see e.g. Eq. (6.15) of the last paper of [10] and see also [25, 26, 27]) in the limit $\Gamma_\omega \rightarrow 0$. The results of Eq. (5.25) show that the final value of the chemical potential depends on the properties of the flow entering the definition of Γ_ω . In summary we can say that the magnetic currents and the vortical currents can affect a number of processes such as the formation of the baryon asymmetry or the dynamics of the electroweak phase transition. Similar kinds of effects can be expected in the case of strongly interacting plasmas where the magnetic gyrotropy can also determine the properties of the chemical potential.

¹⁵There have been a number of suggestions for possible roles that the abelian hypermagnetic Chern-Simons term might play in cosmology. One of them is related to the observation that right-handed electrons, which do not take part in weak interactions and also have a very small Yukawa coupling, are practically decoupled from the thermal ensemble above temperatures of about 10 TeV.

6 Concluding remarks

The dispersion relations of anomalous magnetohydrodynamics are affected by the vortical and the hypermagnetic currents. The vortical currents do not impact on the high-frequency branch of the spectrum but the opposite is true at lower frequencies where new solutions describe the simultaneous presence of hypermagnetic knots and fluid vortices. These parity-odd configurations carry, respectively, hypermagnetic and kinetic gyrotropy. The physical properties of the system roughly interpolate between the features of conventional chiral liquids and the results valid for cold electromagnetic plasmas. While chiral currents are anomalous and do not contribute to entropy production, vector currents are associated with the generalized Joule heating.

When chiral and Ohmic currents are simultaneously present the second law of thermodynamics constrains the kinetic coefficients. The hypermagnetic, vortical and Ohmic currents affect the evolution of the gauge fields and determine the hyperelectric field of the plasma. In anomalous magnetohydrodynamics the perfectly conducting limit is well posed and the hypermagnetic helicity of the knots is strictly conserved, as it happens in the case of conventional plasmas. The hypermagnetic currents are then completely washed out in the perfectly conducting limit and strongly suppressed when the conductivity is large but finite. Close to thermal equilibrium the concentration of the chiral species and the corresponding chemical potential will depend not only on the hypermagnetic gyrotropy but also and on the vortical currents.

In summary the evolution equations of anomalous magnetohydrodynamics offer a minimal theoretical framework where the interplay between conduction currents and chiral currents can be quantitatively analyzed. It is therefore fair to say that the results derived here complement and extend some of the present and earlier strategies aimed at an improved understanding of chiral liquids when generalized Ohmic effects cannot be neglected in the evolution of hypermagnetic and hyperelectric fields at finite fermionic density.

A Generalized Appleton-Hartree equation

A.1 Explicit form of ε_s and ε_v

We are going to give, in what follows the explicit form of the dielectric tensors appearing in section 3. The matrix form of $\varepsilon_s(\Omega)$ and $\varepsilon_v(\Omega)$ is given by:

$$\varepsilon_s(\Omega) = \begin{pmatrix} \varepsilon_1(\Omega) & i\varepsilon_2(\Omega) & 0 \\ -i\varepsilon_2(\Omega) & \varepsilon_1(\Omega) & 0 \\ 0 & 0 & \varepsilon_{\parallel}(\Omega) \end{pmatrix}, \quad \varepsilon_v(\Omega) = \begin{pmatrix} \varepsilon_3(\Omega) & -i\varepsilon_4(\Omega) & 0 \\ i\varepsilon_4(\Omega) & \varepsilon_3(\Omega) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (\text{A.1})$$

where $\varepsilon_1(\Omega)$, $\varepsilon_2(\Omega)$, $\varepsilon_3(\Omega)$, $\varepsilon_4(\Omega)$ and $\varepsilon_{\parallel}(\Omega)$ are defined as:

$$\varepsilon_1(\Omega) = 1 - \frac{\Omega_{p+}^2}{\Omega^2 - \Omega_{B+}^2} - \frac{\Omega_{p-}^2}{\Omega^2 - \Omega_{B-}^2}, \quad (\text{A.2})$$

$$\varepsilon_2(\Omega) = \left(\frac{\Omega_{B-}}{\Omega} \right) \frac{\Omega_{p-}^2}{\Omega^2 - \Omega_{B-}^2} - \left(\frac{\Omega_{B+}}{\Omega} \right) \frac{\Omega_{p+}^2}{\Omega^2 - \Omega_{B+}^2}. \quad (\text{A.3})$$

$$\varepsilon_{\parallel}(\Omega) = 1 - \frac{\Omega_{p+}^2}{\Omega^2} - \frac{\Omega_{p-}^2}{\Omega^2}, \quad (\text{A.4})$$

$$\varepsilon_3(\Omega) = \frac{q\Omega}{(m_+ + m_-)} \left[\frac{1}{\Omega_{B-}^2 - \Omega^2} - \frac{1}{\Omega_{B+}^2 - \Omega^2} \right], \quad (\text{A.5})$$

$$\varepsilon_4(\Omega) = \frac{q}{(m_+ + m_-)} \left[\frac{\Omega_{B-}}{\Omega_{B-}^2 - \Omega^2} + \frac{\Omega_{B+}}{\Omega_{B+}^2 - \Omega^2} \right]. \quad (\text{A.6})$$

Both $\varepsilon_3(\Omega)$ and $\varepsilon_4(\Omega)$ have dimensions of an inverse frequency squared; $\varepsilon_1(\Omega)$, $\varepsilon_2(\Omega)$ and $\varepsilon_{\parallel}(\Omega)$ are instead dimensionless. The frequencies appearing in Eqs. (A.2)–(A.4) and (A.5)–(A.6) are the plasma and the Larmor frequencies associated with the charge carriers of both signs, i.e. $\Omega_{p\pm} = \sqrt{4\pi q^2 n_0}/m_{\pm}$ and $\Omega_{B\pm} = qB_0/m_{\pm}$. To compare the dispersion relations with the standard situation of cold plasmas we must bear in mind that the ratios of the plasma and Larmor frequencies are related to the inverse ratio of the masses, i.e. $\Omega_{p+}/\Omega_{p-} = \Omega_{B+}/\Omega_{B-} = m_-/m_+$.

A.2 The seven function

The generalized form of the Appleton-Hartree equation (see Eq. (4.1)) depends on 7 functions whose explicit form is given by:

$$\begin{aligned} f_B(\varepsilon, \Omega, n, \theta) &= \frac{\varepsilon_{\parallel}}{n^4 \Omega^2} \cos^2 \theta + \frac{(\varepsilon_L + \varepsilon_R)}{2n^4 \Omega^2} \sin^2 \theta, \\ g_B(\varepsilon, \Omega, n, \theta) &= \frac{\varepsilon_{\parallel}(\varepsilon_L - \varepsilon_R)}{n^5 \Omega} \cos \theta, \\ f_{\omega}(\varepsilon, \Omega, n, \theta) &= \frac{\cos^2 \theta}{n^4} \left\{ (\varepsilon_3 - \varepsilon_4)(\varepsilon_3 + \varepsilon_4)\varepsilon_{\parallel} + n^2 \left[\sin \theta \varepsilon_3^2 - (\varepsilon_3^2 - 2\varepsilon_4^2) \sin^2 \theta \right] \right\}, \end{aligned}$$

$$\begin{aligned}
g_\omega(\varepsilon, \Omega, n, \theta) &= \frac{\cos \theta}{2n^5} \left\{ 2\varepsilon_\parallel \left[-n^2\varepsilon_4 + \varepsilon_3(\varepsilon_L - \varepsilon_R) + \varepsilon_4(\varepsilon_L + \varepsilon_R) \right] - 2n^2\varepsilon_4\varepsilon_\parallel \cos^2 \theta \right. \\
&\quad \left. + n^2\varepsilon_3(\varepsilon_L - \varepsilon_R) \sin \theta + n^2 \left[4n^2\varepsilon_4 - 2\varepsilon_3(\varepsilon_L - \varepsilon_R) - 3\varepsilon_4(\varepsilon_L + \varepsilon_R) \right] \sin^2 \theta \right\}, \\
h_1(\varepsilon, \Omega, n, \theta) &= -\frac{2\varepsilon_4}{n^3\Omega^2} \cos \theta \sin^2 \theta, \\
h_2(\varepsilon, \Omega, n, \theta) &= \frac{1}{2n^4\Omega} \left\{ 4\varepsilon_3\varepsilon_\parallel \cos^2 \theta + \sin \theta \left[\varepsilon_3(\varepsilon_L + \varepsilon_R) + \varepsilon_4(-\varepsilon_L + \varepsilon_R) \sin \theta \right] \right\}, \\
h_3(\varepsilon, \Omega, n, \theta) &= -\frac{\varepsilon_3\varepsilon_4 \sin 2\theta(1 + \sin \theta)}{2n^3\Omega}. \tag{A.7}
\end{aligned}$$

The functions reported in Eq. (A.7) determine, through Eq. (4.1), the form of the dispersion relations when the hypermagnetic and the vortical currents are simultaneously present in the anomalous magnetohydrodynamics equations.

B Hypermagnetic knots and Beltrami fields

In the resistive approximation, the hyperelectric and the hypermagnetic fields are not exactly orthogonal and the nature of this misalignment is crucial both for the generation of the baryon asymmetry and for the chiral magnetic effect. In AMHD the induced hyperelectric field stems directly from the approximate form of the Ohm's law and it vanishes exactly, in the plasma frame, when the conductivity goes formally to infinity. In the same limit the contribution of the chemical potential to the anomalous hypermagnetic diffusivity equation gets always erased. At finite conductivity the anomalous contribution can be often rephrased in terms of the magnetic gyrotropy [2] which defines hypermagnetic knot solutions [10].

B.1 Hypermagnetic knots

The configurations minimizing the hypermagnetic energy density with the constraint that the helicity be conserved coincide, in the perfectly conducting limit, with the ones obtainable in ideal magnetohydrodynamics where the anomalous currents are neglected [1, 2, 3, 4].

In the perfectly conducting limit Eq. (5.6) leads to $\partial_t \vec{B} = \vec{\nabla} \times (\vec{v} \times \vec{B}) + \mathcal{O}(\mu_R/\sigma)$ which is qualitatively similar to the result of conventional magnetohydrodynamics. Defining the vector potential in the Coulomb gauge, the magnetic diffusivity equation becomes, up to small corrections, $\partial_t \vec{A} = \vec{v} \times (\vec{\nabla} \times \vec{A})$. The analysis of Ref. [22] can then be exploited. The magnetic energy density shall then be minimized in a finite volume under the assumption of constant magnetic helicity by introducing the Lagrange multiplier p_B . By taking the functional variation of $\mathcal{G} = \int_V d^3x \{ |\vec{\nabla} \times \vec{A}|^2 - p_B \vec{A} \cdot (\vec{\nabla} \times \vec{A}) \}$, with respect to \vec{A} and by requiring $\delta \mathcal{G} = 0$, the configurations extremizing \mathcal{G} are such that $\vec{\nabla} \times \vec{B} = p_B \vec{B}$. In performing the functional variation we assumed that V is the fiducial volume of a closed system.

The configurations $\vec{\nabla} \times \vec{B} = p_B \vec{B}$ have been used to describe hypermagnetic knots (see [10], third and fourth papers); in this case q has dimensions of an inverse length and sets the

scale of the hypermagnetic knot which is related to Chern-Simons waves. The configurations with constant p_B represent the lowest state of magnetic energy which a closed system may attain also in the case where anomalous currents are present, provided the ambient plasma is perfectly conducting.

B.2 Gyrotropic bases

The knotted solutions can be expanded in an appropriate gyrotropic basis. Let us then consider a vector field \vec{a} fields satisfying $\vec{a} \times (\vec{\nabla} \times \vec{a}) = 0$. The simplest realization of these Beltrami fields is provided by the eigenvectors of the curl operator but more general situations are known and have been extensively examined in the literature. Two gyrotropic and orthonormal bases of opposite parity are given by $(\hat{a}, \hat{b}, \hat{z})$ and by $(\hat{c}, \hat{d}, \hat{z})$

$$\hat{a}(z, p) = \{\cos pz, -\sin pz, 0\}, \quad \hat{b}(z, p) = \{\sin pz, \cos pz, 0\}, \quad (\text{B.1})$$

$$\hat{c}(z, p) = \{\cos pz, \sin pz, 0\}, \quad \hat{d}(z, p) = \{-\sin pz, \cos pz, 0\}, \quad (\text{B.2})$$

As anticipated the bases of Eqs. (B.1) and (B.2) are orthonormal. Indeed we have $\hat{a} \cdot \hat{b} = \hat{a} \cdot \hat{z} = \hat{b} \cdot \hat{z} = 0$ and $(\hat{a} \times \hat{b}) \cdot \hat{z} = 1$ (and similarly for \hat{c}, \hat{d} and \hat{z}). The unit vectors of Eqs. (B.1) and (B.2) are normalized eigenvectors of the curl operator with eigenvalues $+p$ and $-p$.

In ordinary MHD knotted solutions can be constructed from Beltrami fields by postulating a solenoidal (static) current and by neglecting the displacement current. In anomalous magnetohydrodynamics these simple constructions cannot be immediately extended because of the magnetic and vortical currents. The knot solutions obtainable by extremizing the functional \mathcal{G} correspond to uniform magnetic fields well inside the core of the knot. This conclusion is evident if we use the basis of Eqs. (B.1) and (B.2). For instance in the limit $pz < 1$ the field configuration $\vec{B}(z, p) = B_0 \hat{a}(z, p) \rightarrow B_0 \hat{x}$ is practically uniform and directed along the \hat{x} axis. The connections between Beltrami fields, force-free solutions in ordinary MHD equilibrium and electromagnetic waves propagation have been explored in a number of papers [22, 23, 20, 21] starting from the classic works of Fermi and Chandrasekhar [28]. It is also possible to obtain hypermagnetic knot solutions with finite helicity and finite gyrotropy which do not satisfy the relation of Beltrami fields. These solutions have been studied in a number of interesting frameworks (see last two papers of Ref. [10] and also [29, 30]).

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